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A note on the velocity of granular flow down a bumpy inclined plane

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Abstract The velocity distribution of granular flow down a bumpy inclined plane is theoretically studied. The characteristic length scale of local transient cluster plays an important role in determining the flow rheology. After discussing the factors influencing the cluster size, we reproduce all observed velocity distributions successfully.

Keywords Granular flow, Cluster, Velocity

Although it seems simple, the granular flow down an inclined plane is not fully understood [1–7]. For example, the observed velocity profiles are different dramatically. Some experimental results [8] are in agreement with the pioneering work of Bagnold [9], in which the velocity profile is a concave function of the distance from the bottom plane. However, convex and linear velocity profiles are also reported [3, 10–14]. The very recent experiments and numerical simulations have shown that all of these velocity profiles could exist [7,15,16]. Despite numerous theoretical works have been devoted to describe the flow properties and a significant progress has been achieved [9,12,17–24], a model which can predict all observed velocity distributions is lacking.

We choose axes such that the flow direction is x, the direction perpendicular to the bumpy plane inclined at an angle θ to the horizontal is y. The free surface and the bumpy plane are $y = h$ and $y = 0$, respectively(see Fig.1). For a steady flow associated with random motions of grains, the time-average velocity or the mean velocity and the shear stress take the following form

$$\bar{u} = \bar{u}(y), \quad \bar{v} = 0 \tag{1}$$

$$\tau = -\rho\overline{u'v'} \tag{2}$$

where u' and v' are velocity fluctuation components, ρ is the density of the granular material.

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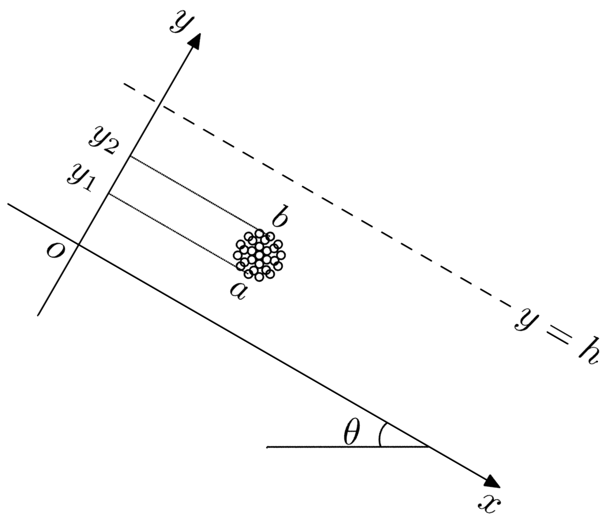

Fig. 1. Schematic diagram of granular flow down a bumpy inclined plane

The fluctuation superimposed on the principal motion is complex in details [3, 6, 11, 25]. While the main variable of interest is the mean velocity. So we have to make some theoretical assumptions for the relation between the fluctuational velocity and the mean velocity. In contrast to the molecules of an ordinary fluid, the collisions between the grains of a granular material are inelastic and lead to the dissipation of energy. This effect tends to aggregate grains together and form clusters [26]. These spatial structures are well known in granular gasses [27–29]. Indeed, they have also been observed in dense granular flows [6, 30]. Many previous studies reveal that the *micro-structures* in the flowing layer play an important role in determining the flow rheology. For example, Mills *et al* assumed the *strong contact networks* and the *weak contact networks* coexist in a transient way [18]; Etraş and Halsey proposed the formation of *granular eddies*, large scale structures of grains moving coherently with one another [21]. In fact, the clusters are neither chainlike nor the whole packing, but rather fractional [6]. They can unite, split, emerge, and disappear.

It is convenient to assume that the flowing layer is composed of transient clusters. The velocity fluctuations originate from the grain transportation between the local transient cluster and its neighbors. We expect that two velocity fluctuation components have the same order. Consider a cluster as shown in Fig.1, when it rolls, the boundary grains are more easier to leave this cluster due to the centrifugal effect and the impacts of surrounding grains. The maximum displacement, a grain belonging to this cluster can arrive in the y direction, is the distance between a and b. The difference of the mean velocity between a and b can be regarded as the velocity fluctuation u' at a. Neglecting all high-order terms in a Taylor series, we obtain

$$\overline{|u'|}_a = |\bar{u}_b - \bar{u}_a| \approx (y_2 - y_1)|\frac{d\bar{u}}{dy}|_a = l|\frac{d\bar{u}}{dy}|_a \tag{3}$$

where l is a unknown length scale.

Note that it is positive within the whole layer, the shear stress can be written as

$$\tau = \rho l^2 (\frac{d\bar{u}}{dy})^2 \tag{4}$$

where all related constants have been included with l.

Comparing Eq. (4) with Prandtl's mixing-length hypothesis [31], we find that l is analogous with the mixing-length. It can be regarded as the *granular mixing-length*. Different from Prandtl's mixing-length, the granular mixing-length is a measure of characteristic length scale of the local transient cluster. It should be pointed out that although using Bagnold model directly, a like ideal has appeared in the works of Khakhar *et al* [32] in which the granular mixing-length is interpreted as a measure of inter-grain spacing. Applying dimensional analysis, Etraş and Halsey [21] defined an effective *viscosity length scale* and also gave a similar expression of Eq. (4).

The momentum balance equation and boundary conditions are as follows

$$\rho g \sin\theta + \frac{d\tau}{dy} = 0 \tag{5}$$

$$\tau|_{y=h} = 0, \quad \bar{u}|_{y=0} = 0 \tag{6}$$

To solve Eq. (5), the relation between the granular mixing-length, l, and the spatial coordinate, y, and other important parameters must be founded. Now, let's discuss the factors influencing the cluster size in this system. Pouliquen [2] has shown experimentally that the critical thickness, $h_{stop}(\theta)$, remaining when the flow stops, is a characteristic length scale. It is conceivable that all grains within the whole depth form a large cluster when the flow stops. So $h_{stop}(\theta)$ reflects the maximum size a cluster can arrive. It contains the effects of the inclination, θ, and the bumpy plane. Of course, the grain diameter, d, is another important parameter. For different non-dimensional depths, $\lambda = \frac{h}{d}$, the influences of these parameters are different.

(a) For thick flowing layer, the cluster size is much larger than d. There is a typical size [6], which is equal to $h_{stop}(\theta)$ in the model of Etraş and Halsey [21]. We assume that the granular mixing-length is proportional to $h_{stop}(\theta)$

$$l = A_1 h_{stop}(\theta) \tag{7}$$

In this case, we get Bagnold profile

$$\bar{u} = \frac{2h\sqrt{gh\sin\theta}}{3A_1 h_{stop}(\theta)}[1 - (1 - \frac{y}{h})^{\frac{3}{2}}] \tag{8}$$

Comparing with Pouliquen scaling relation [2]

$$\frac{U}{\sqrt{gh}} = \beta \frac{h}{h_{stop}(\theta)} \tag{9}$$

where U is the depth average velocity, we find that

$$A_1 = \frac{2}{5\beta}\sqrt{\sin\theta} \tag{10}$$

where $\beta = 0.136$ [2]. Silbert *et al* [7] obtained $\beta = 0.147$.

(b) For medium flowing layer, the effect of grain size can't be neglected. The numerical simulations of driven granular gases with gravity have shown that the cluster size increases with the grain size and the distance from the free surface [27]. In the study of the granular flow in a rotating cylinder, Khakhar *et al* [32] made the following assumption

$$l = C_1\sqrt{d(h-y)} \tag{11}$$

We further assume that the maximum granular mixing-length is related to $h_{stop}(\theta)$ through

$$l_{max} = A_2 h_{stop}(\theta) \tag{12}$$

Using the same method as in (a), we find a linear velocity profile

$$\bar{u} = \frac{h\sqrt{gh\sin\theta}}{A_2 h_{stop}(\theta)}\frac{y}{h} \tag{13}$$

where

$$A_2 = \frac{1}{2\beta}\sqrt{\sin\theta} \tag{14}$$

(c) For thin flowing layer, the effect of the bumpy plane is very strong. The cluster size must increase more rapidly than that of case (b), when approaching the bumpy plane. This means that the power of $(h-y)$ in the expression of l is greater than $\frac{1}{2}$. Since l has a length dimension, a suitable selection is

$$l = C_2(h-y) \tag{15}$$

The restriction of l still is

$$l_{max} = A_3 h_{stop}(\theta) \tag{16}$$

Thus, we obtain a convex velocity profile(in other words, the velocity gradient will increase as one approaches the surface)

$$\bar{u} = \frac{2h\sqrt{gh\sin\theta}}{A_3 h_{stop}(\theta)}[1 - (1 - \frac{y}{h})^{\frac{1}{2}}] \tag{17}$$

where

$$A_3 = \frac{2}{3\beta}\sqrt{\sin\theta} \tag{18}$$

We need know two critical points of λ, at which the transition of the shape of velocity profile occurs, to predict the actual velocity profiles. Sibert *et al* [7,15] observed that Bagnold profile is true only for $\lambda \gtrsim \lambda_c = 20$. Unfortunately, we can not determine the other critical point clearly. Sibert *et al* [7] have shown that the velocity profile is approximately linear when $5 \lesssim \lambda \lesssim 15$. However, the convex velocity profiles have also been investigated at $\lambda \approx 5, 8.1, 11, 14$, respectively [11,13,16]. These results indicate that the flow dynamic is more sensitive to boundary conditions when $\lambda < \lambda_c$. In particular, the variations in the roughness of the bottom plate can lead to very complex flow behavior [33]. Eq. (13) can reproduce the observed profile of Savage *et al* [10]. The predicts of Eq. (17) rather than Eq. (13) are in good agreement with the experiment data of Ancey [16], in which the influence of bumpy plane is stronger.

In summary, the velocity of granular flow down a bumpy inclined plane is mainly determined by the local cluster size or the granular mixing-length. The detailed expressions of the granular mixing-length are discussed in different conditions. Although very crude, the present work reproduces all observed velocity profiles successfully. It seems that the spatial correlations of the velocity field play an important role in dense granular flows.

References

1. A. Daerr & S. Douady, Nature **399** (1999), p. 241
2. O. Pouliquen, Phys. Fluids **11** (1999), p. 542
3. P. A. Lemieux & D. J. Durian, Phys. Rev. Lett. **85** (2000), p. 4273
4. B. Andreotti, A. Daerr & S. Douady, Phys. Fluids **14** (2002), p. 415
5. L. Bocquet, J. Errami & T. C. Lubensky, Phys. Rev. Lett. **89** (2002), 184301
6. D. Bonamy, F. Daviaud, L. Laurent, M. Bonetti & J. P. Bouchaud, Phys. Rev. Lett. **89** (2002), 034301
7. L. E. Silbert, J. W. Landry & G. S. Grest, Phys. Fluids **15** (2003), p. 1
8. S. B. Savege, Adv. Appl. Mech. **24**, 289(1984); N. P. Kruyt, W. J. T. Verel, Powder Tech. **73** (1992), p. 109
9. R. A. Bagnold, Proc. Roy. Soc. London A **225** (1954), p. 49
10. S. B. Savege, R. M. Neddermann, U. Tuzün & G. T. Houlsby, Chem. Eng. Sci. **38** (1983), p. 189
11. D. M. Hanes & O. R. Walton, Powder Tech. **109** (2000), p. 133
12. B. Andreotti & S. Douady, Phys. Rev. E **63** (2001), 031305
13. A. C. Santomso & P. Canu, Chem. Eng. Sci. **56** (2001), p. 3563
14. P. C. Johnson, P. Nott, R. Jackson & J. Fluid Mech. **210** (1990), p. 501
15. L. E. Silbert, D. Ertaş, G. S. Grest, T. C. Halsey, D. Levine & S. J. Plimpton, Phys. Rev. E **64** (2001), 051302
16. C. Ancey, Phys. Rev. E **65** (2002), 011304
17. K. G. Anderson & R. Jackson, J. Fluid Mech. **241** (1992), p. 145
18. P. Mills, D. Loggia & M. Tixier, Europhys. Lett. **45** (1999), p. 733; P. Mills, M. Tixier, D. Loggia, Eur. Phys. J. E **1** (2000), p. 5
19. Y. Q. Wang & K. Hutter, Rheol. Acta **38** (1999), p. 214
20. C. Ancey & P. Evesque, Phys. Rev. E **62** (2000), p. 8349
21. D. Ertaş & T. C. Halsey, Europhys. Lett. **60** (2002), p. 931
22. I. S. Aranson & L. S. Tsimring, Phys. Rev. E **65** (2002), 061303
23. L. Bocquet, W. Losert, D. Schalk, T. C. Lubensky & J. P. Gollub, Phys. Rev. E **65** (2002), 011307
24. M. Y. Louge, Phys. Rev. E **67** (2003), 061303
25. H. Ahn, C. E. Brennen & R. H. Sabersky, ASME J. Appl. Mech. **58** (1991), p. 792
26. H. M. Jaeger, S. R. Nagel & R. P. Behringer, Rev. Mod. Phys. **68** (1996), p. 1259
27. A. Baldassarri, U. Marini Bettolo Marconi, A. Puglisi & A. Vulpiani, Phys. Rev. E **64** (2001), 011301
28. R. Mikkelsen, D. van der Meer, K. van der Weele & D. Lohse, Phys. Rev. Lett. **89** (2002), 214301
29. S. A. Hill & G. F. Mazenko, Phys. Rev. E **67** (2003), 061302
30. W. Polashenski, P. Zamankhan, S. Mäkiharju & P. Zamankhan, Phys. Rev. E **66** (2002), 021303; A. Ferguson, B. Fisher, B. Chakraborty, arXiv:cond-mat (2003), 0301201
31. H. Schlichting, *Boundary-Layer Theory*(McGraw-Hill (1968)), p. 527 and p. 545-549
32. D. V. Khakhar, J. J. McCarthy, T. Shinbrot & J. M. Ottino, Phys. Fluids **9** (1997), p. 31; A. V. Orpe, D. V. Khakhar, Phys. Rev. E **64** (2001), 031302
33. L. E. Silbert, G. S. Grest & S. J. Plimpton, Phys. Fluids **14** (2002), p. 2637